

# Poisson geometry of the Maxwell-Bloch top system and stability problem

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**Abstract.** Dynamics of Maxwell-Bloch top system, that includes Maxwell-Bloch and Lorenz-Hamilton equations as particular cases, is studied in the framework Poisson geometry. Constants of motion as well as the relation of solution to that of pendulum are presented. Equilibrium states are determined and, their complete stability analysis are performed. Results are applied to an optimal control problem on the Lie group  $G_4$ .<sup>1</sup>

## 1 Introduction

The Hamilton-Poisson systems appear naturally in many areas of physical science and engineering including theoretical mechanics of fluids, spatial dynamics and many others [1, 9, 2]. A remarkable class of Hamilton-Poisson systems is formed by a family of differential equations on  $\mathbf{R}^3$  which depend by a triple of real parameters, called the Maxwell-Bloch top system. For certain values of these parameters various integrable systems, such as the real-valued Maxwell-Boch equations [9], Lorenz-Hamilton system [5], etc., are obtained. We shall show that the solution of optimal control problem for left invariant systems on certain matrix Lie groups leads to systems of differential equations belonging to the family of Maxwell-Bloch top.

This paper is structured as follows. In Section 2, we introduce the Maxwell-Bloch top system (2.1) and some dynamical properties of it are established. Also, we show the relation between solution of the Maxwell-Bloch top and that of a pendulum. In Section 3, we investigate Maxwell-Bloch top system in terms of Poisson geometry. Section 4 is dedicated to study of Lyapunov stability for equilibrium states of Maxwell-Bloch top system. In Section 5, we apply results of Sections 2-4 for an optimal control problem of a particular drift-free left invariant system on the special nilpotent four-dimensional Lie group  $G_4$ .

## 2 Dynamical properties of the Maxwell-Bloch top system

Consider the following family of differential equations of Maxwell-Bloch type on  $\mathbf{R}^3$ :

$$\dot{x}_1(t) = b_1 x_2(t), \quad \dot{x}_2(t) = b_2 x_1(t) x_3(t), \quad \dot{x}_3(t) = b_3 x_1(t) x_2(t), \quad (2.1)$$

where  $\dot{x}_i = dx_i(t)/dt$ ,  $i = 1, 2, 3$  and  $b_1, b_2, b_3 \in \mathbf{R}$  are parameters such that  $b_1 b_2 b_3 \neq 0$  and  $t$  is the time. We will refer to the dynamical system (2.1) as the *Maxwell-Bloch top system* and denote the vector of parameters by  $b = (b_1, b_2, b_3)$ .

If in (2.1), we take  $b = (1, 1, -1)$ , then we obtain the *three-dimensional real-valued Maxwell-Bloch equations* [3], given by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 x_3, \quad \dot{x}_3 = -x_1 x_2. \quad (2.2)$$

Also, for  $b = (1/2, -1, 1)$  we obtain the *Lorenz-Hamilton system* [2, 5], given by

<sup>1</sup>Mathematical Subject Classification(2010):34H05, 37C20, 37C75

*Keywords and phrases:* Hamiltonian dynamics, Maxwell-Bloch top system, Lyapunov stability.

$$\dot{x}_1 = \frac{1}{2}x_2, \quad \dot{x}_2 = -x_1x_3, \quad \dot{x}_3 = x_1x_2. \quad (2.3)$$

**Proposition 2.1.** *The functions  $H^b, C^b \in C^\infty(\mathbf{R}^3, \mathbf{R})$  given by:*

$$H^b(x_1, x_2, x_3) = \frac{b_1}{2} \left( x_2^2 - \frac{b_2}{b_3} x_3^2 \right) \quad \text{and} \quad C^b(x_1, x_2, x_3) = -\frac{b_3}{2b_1} x_1^2 + x_3 \quad (2.4)$$

*are constants of the motion (first integrals) for the dynamics (2.1).*

*Proof.* Indeed,

$$\begin{aligned} \frac{dH^b}{dt} &= b_1(x_2\dot{x}_2 - \frac{b_2}{b_3}x_3\dot{x}_3) = b_1(-x_2(b_2x_1x_3) + \frac{b_2}{b_3}x_3(b_3x_1x_2)) = 0 \quad \text{and} \\ \frac{dC^b}{dt} &= -\frac{b_3}{b_1}x_1\dot{x}_1 + \dot{x}_3 = -\frac{b_3}{b_1}x_1(b_1x_2) + b_3x_1x_2 = 0. \end{aligned} \quad \square$$

**Remark 2.1.** From Proposition 2.1 it follows that *the trajectories of the dynamics (2.1) in the phase space  $\mathbf{R}^3$  are the intersections of the surfaces:*

$$b_1x_2^2 - \frac{b_1b_2}{b_3}x_3^2 = 2H^b, \quad -\frac{b_3}{b_1}x_1^2 + 2x_3 = 2C^b,$$

*where  $H^b = \text{constant}$  and  $C^b = \text{constant}$ .*  $\square$

Using the fact that  $H^b$  given by (2.4) is a first integral (see Proposition 2.1) one easily prove that the Maxwell-Bloch top system has the following first integral:

$$H_0^b(x) = \frac{1}{2} \left( x_2^2 - \frac{b_2}{b_3} x_3^2 \right). \quad (2.5)$$

We shall prove that in certain restrictions on  $b_i$ , the motion of Maxwell-Bloch top system reduces to motion on the surface described by the conservation law (2.5).

**Proposition 2.2.** *We assume that  $b_2b_3 < 0$ . The solution of the Maxwell-Bloch top system (2.1) restricted to the constant level surface defined by:*

$$x_2^2 - \frac{b_2}{b_3} x_3^2 = 2H = \text{constant}, \quad H = H_0^b > 0 \quad (2.6)$$

*is*

$$\begin{cases} x_1(t) &= \frac{\gamma}{b_3} \cdot \dot{\theta}(t) \\ x_2(t) &= \sqrt{2H} \cdot \cos \theta(t) \\ x_3(t) &= \gamma \sqrt{2H} \cdot \sin \theta(t) \end{cases} \quad \text{with } \gamma = \sqrt{-\frac{b_3}{b_2}} \quad (2.7)$$

*where  $\theta(t)$  is a solution of the pendulum equation:*

$$\ddot{\theta}(t) = \frac{b_1b_3}{\gamma} \sqrt{2H} \cdot \cos \theta(t). \quad (2.8)$$

*Proof.* Denote  $\gamma = \sqrt{-b_3/b_2} > 0$ . By a direct computation, it is easy to see that

$$(i) \quad x_2(t) = \sqrt{2H} \cdot \cos \theta(t), \quad x_3(t) = \gamma \sqrt{2H} \cdot \sin \theta(t)$$

are solutions of the equation (2.6). By deriving of the second relation of (i) with respect to  $t$ , we have  $\dot{x}_3(t) = \gamma \sqrt{2H} \cdot \cos \theta(t) \cdot \dot{\theta}(t)$  and using the first relation of (i), we obtain:

$$(ii) \quad \dot{x}_3(t) = \gamma \cdot x_2(t) \cdot \dot{\theta}(t).$$

From (ii) and  $\dot{x}_3(t) = b_3 x_1(t)x_2(t)$ , we deduce  $x_1(t) = \frac{\gamma}{b_3} \cdot \dot{\theta}(t)$ . Therefore, the relations (2.7) are verified. From the last equality follows:

$$(iii) \quad \dot{\theta}(t) = \frac{b_3}{\gamma} \cdot x_1(t).$$

Differentiating again the relation (iii) and using the first equation from (2.1) and (i), it follows  $\ddot{\theta}(t) = \frac{b_3}{\gamma} \cdot \dot{x}_1(t) = \frac{b_1 b_3}{\gamma} \sqrt{2H} \cdot \cos \theta(t)$ , i.e. (2.8) holds.  $\square$

**Corollary 2.1.** *The solution of the Lorenz-Hamilton system (2.3), restricted to the constant level surface defined by:*

$$x_2^2 + x_3^2 = 2H = \text{constant}, \quad H > 0$$

is

$$x_1(t) = \dot{\theta}(t), \quad x_2(t) = \sqrt{2H} \cdot \cos \theta(t), \quad x_3(t) = \sqrt{2H} \cdot \sin \theta(t), \quad (2.9)$$

where  $\theta(t)$  is a solution of the pendulum equation:

$$\ddot{\theta}(t) = \frac{1}{2} \sqrt{2H} \cdot \cos \theta(t). \quad (2.10)$$

*Proof.* In Proposition 2.2 we take  $b_1 = 1/2$ ,  $b_2 = -1$ ,  $b_3 = 1$  and we obtain the required result.  $\square$

### 3 Realizations Hamilton-Poisson for the Maxwell-Bloch top system

For definitions and results on Poisson geometry and Hamiltonian dynamics see [9, 2].

**Proposition 3.1.** (i) *The Maxwell-Bloch top system (2.1) is a Hamilton-Poisson system with the phase space  $\mathbf{R}^3$ , the Hamiltonian  $H^b$  given by (2.4) and with respect the Poisson structure  $\{\cdot, \cdot\}$  given by*

$$\{f, g\} = \det \begin{pmatrix} -\frac{b_3}{b_1} x_1 & 0 & 1 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \end{pmatrix}, \quad \text{for all } f, g \in C^\infty(\mathbf{R}^3). \quad (3.1)$$

(ii) *The function  $C^b$  given by (2.4) is a Casimir of the configuration  $(\mathbf{R}^3, \{\cdot, \cdot\})$ .*

*Proof.* (i) It is easy to observe that  $\{f, g\} = \nabla C^b \cdot (\nabla f \times \nabla g)$ . Then  $\{\cdot, \cdot\}$  is a bracket operation on  $\mathbf{R}^3$ .

The system (2.1) is a Hamilton-Poisson system, since  $\dot{x}_i = \{x_i, H^b\}$ ,  $i = 1, 2, 3$ .

Indeed, for example

$$\{x_1, H^b\} = \begin{vmatrix} -\frac{b_3}{b_1} x_1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & b_1 x_2 & -\frac{b_1 b_2}{b_3} x_3 \end{vmatrix} = b_1 x_2 = \dot{x}_1.$$

(ii) The function  $C^b \in C^\infty(\mathbf{R}^3, \mathbf{R})$  is a Casimir, since  $\{C^b, f\} = 0$  for every  $f \in C^\infty(\mathbf{R}^3, \mathbf{R})$ . We have  $\{C^b, f\} = \nabla C^b \cdot (\nabla C^b \times \nabla f) = 0$ .  $\square$

We can easily prove that the Poisson structure  $\{\cdot, \cdot\}$  given by (3.1) is in fact generated by the skew-symmetric matrix

$$P^b(x_1, x_2, x_3) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{b_3}{b_1}x_1 \\ 0 & \frac{b_3}{b_1}x_1 & 0 \end{pmatrix}. \quad (3.2)$$

The Maxwell-Bloch top system (2.1) can be expressed in the matrix form:

$$\dot{X} = P^b(x) \cdot \nabla H(x), \quad (3.3)$$

where  $x = (x_1, x_2, x_3)$  and  $\dot{X} = (\dot{x}_1 \ \dot{x}_2 \ \dot{x}_3)^T$ .

Define the functions  $C_{\alpha\beta}^b, H_{\gamma\delta}^b \in C^\infty(\mathbf{R}^3, \mathbf{R})$  be given by:

$$C_{\alpha\beta}^b = \alpha C^b + \beta H^b, \quad H_{\gamma\delta}^b = \gamma C^b + \delta H^b, \quad \alpha, \beta, \gamma, \delta \in \mathbf{R} \quad \text{that is} \quad (3.4)$$

$$\begin{cases} C_{\alpha\beta}^b(x_1, x_2, x_3) &= -\frac{\alpha b_3}{2b_1}x_1^2 + \frac{\beta b_1}{2}x_2^2 + \alpha x_3 - \frac{\beta b_1 b_2}{2b_3}x_3^2 \\ H_{\gamma\delta}^b(x_1, x_2, x_3) &= -\frac{\gamma b_3}{2b_1}x_1^2 + \frac{\delta b_1}{2}x_2^2 + \gamma x_3 - \frac{\delta b_1 b_2}{2b_3}x_3^2 \end{cases} \quad (3.5)$$

**Proposition 3.2.** (i) The Maxwell-Bloch top system (2.1) admits a family of Hamilton-Poisson realizations parametrized by the group  $SL(2; \mathbf{R})$ . More precisely,  $(\mathbf{R}^3, \{\cdot, \cdot\}_{\alpha\beta}^b, H_{\gamma\delta}^b)$  is a Hamilton-Poisson realization of the system (2.1), where  $\{\cdot, \cdot\}_{\alpha\beta}$  is given by:

$$\{f, g\}_{\alpha\beta}^b = \det \begin{pmatrix} -\frac{\alpha b_3}{b_1}x_1 & \beta b_1 x_2 & \alpha - \frac{\beta b_1 b_2}{b_3}x_3 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \end{pmatrix}, \quad \forall f, g \in C^\infty(\mathbf{R}^3, \mathbf{R}), \quad (3.6)$$

the Hamiltonian  $H_{\gamma\delta}^b$  is given by (3.5) and the matrix  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2; \mathbf{R})$ .

(ii)  $C_{\alpha\beta}^b$  given by (3.5) is a Casimir of the configuration  $(\mathbf{R}^3, \{\cdot, \cdot\}_{\alpha\beta}^b)$ .

*Proof.* (i) We have  $\frac{\partial H_{\gamma\delta}^b}{\partial x_1} = \frac{\gamma b_3}{b_1}x_1$ ,  $\frac{\partial H_{\gamma\delta}^b}{\partial x_2} = \delta b_1 x_2$ ,  $\frac{\partial H_{\gamma\delta}^b}{\partial x_3} = \gamma - \frac{\delta b_1 b_2}{b_3}x_3$ . Then:

$$\{x_1, H_{\gamma\delta}^b\}_{\alpha\beta}^b = \det \begin{pmatrix} -\frac{\alpha b_3}{b_1}x_1 & \beta b_1 x_2 & \alpha - \frac{\beta b_1 b_2}{b_3}x_3 \\ 1 & 0 & 0 \\ \frac{\gamma b_3}{b_1}x_1 & \delta b_1 x_2 & \gamma - \frac{\delta b_1 b_2}{b_3}x_3 \end{pmatrix} = (\alpha\delta - \beta\gamma)b_1 x_2 = \dot{x}_1.$$

Similarly, we have  $\{x_2, H_{\gamma\delta}^b\}_{\alpha\beta}^b = b_2 x_1 x_3 = \dot{x}_2$  and  $\{x_3, H_{\gamma\delta}^b\}_{\alpha\beta}^b = b_3 x_1 x_2 = \dot{x}_3$ . Therefore, one obtains the required result.

(ii) It is easy to see that  $\{C_{\alpha\beta}^b, f\}_{\alpha\beta}^b = 0$ , for all  $f \in C^\infty(\mathbf{R}^3, \mathbf{R})$ .  $\square$

The Poisson structure given by (3.6) is generated by the matrix

$$P_{\alpha\beta}^b(x_1, x_2, x_3) = \begin{pmatrix} 0 & \alpha - \frac{\beta b_1 b_2}{b_3} x_3 & -\beta b_1 x_2 \\ -\alpha + \frac{\beta b_1 b_2}{b_3} x_3 & 0 & -\frac{\alpha b_3}{b_1} x_1 \\ \beta b_1 x_2 & \frac{\alpha b_3}{b_1} x_1 & 0 \end{pmatrix}. \quad (3.7)$$

**Remark 3.1.** Proposition 3.2 assures that the equations (2.1) are invariant, if  $H^b$  and  $C^b$  are replaced by linear combinations with coefficients modulo  $SL(2, \mathbf{R})$ . In consequence, the trajectories of motion of the system (2.1) remain unchanged.  $\square$

Finally, we can conclude that *the Maxwell-Bloch top system (2.1) has the following Hamilton-Poisson realization*

$$(\mathbf{R}^3, P_{\alpha\beta}^b, H_{\gamma\delta}^b) \quad \text{with Casimir } C_{\alpha\beta}^b, \quad (3.8)$$

where  $P_{\alpha\beta}^b$  is given by (3.7) and  $H_{\gamma\delta}^b$ ,  $C_{\alpha\beta}^b$  are given by (3.5) for all  $\alpha, \beta, \gamma, \delta \in \mathbf{R}$  such that  $\alpha\delta - \beta\gamma = 1$ .

If in (3.8) we take  $\alpha = 1, \beta = \gamma = 0$  and  $\delta = 1$ , then one obtains Proposition 3.1. More precisely,  $(\mathbf{R}^3, P^b, H^b)$  is a Hamilton-Poisson realization of the dynamics (2.1) with Casimir  $C^b$ , since  $H_{01}^b = H^b$ ,  $C_{10}^b = C^b$  and  $P_{10}^b = P^b$ .

Next proposition gives another special Hamilton-Poisson realization of the Maxwell-Bloch top system.

**Proposition 3.3.** (i) *The Maxwell-Bloch top system (2.1) has the Hamilton-Poisson realization  $(\mathbf{R}^3, \bar{P}^b, \bar{H}^b)$ , where the matrix  $\bar{P}^b$  is given by*

$$\bar{P}^b(x_1, x_2, x_3) = \begin{pmatrix} 0 & \frac{b_1 b_2}{b_3} x_3 & b_1 x_2 \\ -\frac{b_1 b_2}{b_3} x_3 & 0 & 0 \\ -b_1 x_2 & 0 & 0 \end{pmatrix}, \quad (3.9)$$

and the Hamiltonian  $\bar{H}^b$  is given by

$$\bar{H}^b(x_1, x_2, x_3) = -\frac{b_3}{2b_1} x_1^2 + x_3. \quad (3.10)$$

(ii) *The function  $\bar{C}^b$  defined by*

$$\bar{C}^b(x_1, x_2, x_3) = \frac{b_1}{2} \left( x_2^2 - \frac{b_2}{b_3} x_3^2 \right). \quad (3.11)$$

is a Casimir of the configuration  $(\mathbf{R}^3, \{\cdot, \cdot\}_1)$ , where  $\{\cdot, \cdot\}_1$  is the bracket operation whose its Poisson matrix is  $\bar{P}^b$ .

*Proof.* The assertions are consequences of Proposition 3.2. For  $\beta = -1, \alpha = \delta = 0$  and  $\gamma = 1$  one obtains  $\{\cdot, \cdot\}_1 = \{\cdot, \cdot\}_{0,-1}^b, \bar{H}^b = H_{10}^b, \bar{C}^b = -C_{0,-1}^b$  and  $\bar{P}^b = P_{0,-1}^b$ .  $\square$

Using Propositions 3.2 and 3.3 we obtain the following proposition.

**Proposition 3.4.** *The Maxwell-Bloch top system (2.1) have the following two (special) Hamilton-Poisson realizations:*

(i)  $(\mathbf{R}^3, P^b, H^b)$  with the Casimir  $C^b \in C^\infty(\mathbf{R}^3, \mathbf{R})$ , where  $P^b$  is given by (3.2) and  $H^b, C^b$  are given by (2.4);

(ii)  $(\mathbf{R}^3, \bar{P}^b, \bar{H}^b)$  with the Casimir  $\bar{C}^b \in C^\infty(\mathbf{R}^3, \mathbf{R})$ , where  $\bar{P}^b$  is given by (3.9) and  $\bar{H}^b, \bar{C}^b$  are given by (3.10) and (3.11), respectively.  $\square$

**Remark 3.2.** We have  $\bar{P}^b = \Pi_{(0,u,v)}$  with  $u = b_1, v = -b_1 b_2 / b_3$  (see the relation (2.5)

in [6]), where  $\Pi_{(0,u,v)} = \begin{pmatrix} 0 & -vx_3 & ux_2 \\ vx_3 & 0 & 0 \\ -ux_2 & 0 & 0 \end{pmatrix}$ . Hence the Poisson geometry of the system (2.1) is generated by a matrix of  $se(2)$ -type (here,  $se(2)$  is the Lie algebra of the Lie group  $SE(2; \mathbf{R})$ ).  $\square$

**Remark 3.3.** Applying Proposition 3.4 one obtains two special Hamilton-Poisson realizations for the real Maxwell-Bloch equations (2.2) and Lorenz-Hamilton system (2.3), respectively.  $\square$

## 4 Stability problem for Maxwell-Bloch top dynamics

A direct computation shows that the equilibrium states of the Maxwell-Bloch top system (2.1) are the points

$$e_0 = (0, 0, 0), \quad e_1^m = (m, 0, 0) \quad \text{and} \quad e_3^m = (0, 0, m) \quad \text{for all } m \in \mathbf{R}^*.$$

Let  $A(x_1, x_2, x_3)$  be the matrix of the linearisation of the system (2.1), i.e.

$$A(x_1, x_2, x_3) = \begin{pmatrix} 0 & b_1 & 0 \\ b_2 x_3 & 0 & b_2 x_1 \\ b_3 x_2 & b_3 x_1 & 0 \end{pmatrix}.$$

**Proposition 4.1.** (i) *The equilibrium states  $e_1^m$ ,  $m \in \mathbf{R}^*$  are spectrally stable if  $b_2 b_3 < 0$  and unstable if  $b_2 b_3 > 0$ .*

(ii) *The equilibrium states  $e_3^m$ ,  $m \in \mathbf{R}^*$  are spectrally stable if  $mb_1 b_2 < 0$  and unstable if  $mb_1 b_2 > 0$ .*

(iii) *The equilibrium state  $e_0 = (0, 0, 0)$  is spectrally stable.*

*Proof.* (i) The characteristic polynomial of

$$A(e_1^m) = \begin{pmatrix} 0 & b_1 & 0 \\ 0 & 0 & mb_2 \\ 0 & mb_3 & 0 \end{pmatrix}$$

is  $p_{A(e_1^m)}(\lambda) = \det(A(e_1^m) - \lambda I) = -\lambda(\lambda^2 - b_2 b_3 m^2)$ . Then the characteristic roots of  $A(e_1^m)$  are  $\lambda_1 = 0$  and  $\lambda_{2,3} = \pm m \sqrt{b_2 b_3}$ , if  $b_2 b_3 > 0$  and  $\lambda_{2,3} = \pm im \sqrt{-b_2 b_3}$ , if  $b_2 b_3 < 0$ . From Lyapunov's Theorem it follows that  $e_1^m$  is spectrally stable for  $b_2 b_3 < 0$  and unstable for  $b_2 b_3 > 0$ .

(ii) The characteristic polynomial of

$$A(e_3^m) = \begin{pmatrix} 0 & b_1 & 0 \\ mb_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is  $p_{A(e_3^m)}(\lambda) = -\lambda(\lambda^2 - mb_1b_2)$  with the characteristic roots  $\lambda_1 = 0$ ,  $\lambda_{2,3} = \pm\sqrt{mb_1b_2}$ . Applying now similar arguments as in the proof of the assertion (i), one obtains the required results.

(iii) It is easy to see that  $e_0$  is spectrally stable.  $\square$

Let us discuss the nonlinear stability of equilibrium states of the dynamics (2.1) which are spectrally stable. Recall that an equilibrium state  $x_e$  is nonlinear stable if the trajectories starting close to  $x_e$  stay close to  $x_e$  (i.e. a neighborhood of  $x_e$  must be flow invariant).

**Proposition 4.2.** *If  $b_2b_3 < 0$ , then  $e_1^m$ ,  $m \in \mathbf{R}^*$  is nonlinear stable.*

*Proof.* We suppose that  $b_2b_3 < 0$ . We shall make the proof using Lyapunov's theorem [4]. Let be the function  $L^\alpha : \mathbf{R}^3 \rightarrow \mathbf{R}$  given by:

$$L^b(x_1, x_2, x_3) = \frac{1}{2} \left( -\frac{b_3}{2b_1}x_1^2 + x_3 + \frac{b_3}{2b_1}m^2 \right)^2 + \frac{1}{2} \left( x_2^2 - \frac{b_2}{b_3}x_3^2 \right).$$

For the function  $L^b$  we have successively:

(i)  $L^b \in C^\infty(\mathbf{R}^3, \mathbf{R})$  and  $L^b(m, 0, 0) = 0$ ;

(ii)  $L^b(x_1, x_2, x_3) > 0$ , for all  $x \in \mathbf{R}^3$ ,  $x \neq e_1^m$ , since  $b_2b_3 < 0$ ;

(iii) The derivative of  $L^b$  with respect to  $t$  along the trajectories of the dynamics (2.4) is zero. Indeed,

$$\begin{aligned} \frac{dL^b}{dt} &= \frac{\partial L^b}{\partial x_1} \dot{x}_1 + \frac{\partial L^b}{\partial x_2} \dot{x}_2 + \frac{\partial L^b}{\partial x_3} \dot{x}_3 = -\frac{b_3}{b_1}x_1 \left( -\frac{b_3}{2b_1}x_1^2 + x_3 + \frac{b_3}{2b_1}m^2 \right) b_1x_2 + x_2(b_2x_1x_3) + \\ &+ \left[ \left( -\frac{b_3}{2b_1}x_1^2 + x_3 + \frac{b_3}{2b_1}m^2 \right) - \frac{b_2}{b_3}x_3 \right] (b_3x_1x_2) = 0. \end{aligned}$$

Therefore  $L^b$  is a Lyapunov function. Then via Lyapunov's theorem we obtain that  $e_1^m$  is nonlinear stable.  $\square$

**Proposition 4.3.** *The equilibrium state  $e_0$  of the dynamics (2.1) is nonlinear stable.*

*Proof.* An easy computation shows that

$$L_0^b(x_1, x_2, x_3) = \frac{1}{4} \left( x_2^2 - \frac{b_2}{b_3}x_3^2 \right)^2$$

is a Lyapunov function. The assertion is a consequence of the Lyapunov theorem.  $\square$

**Proposition 4.4.** *If  $mb_1b_2 < 0$ , then  $e_3^m$  is nonlinear stable.*

*Proof.* We shall make the proof using Arnold's energy-Casimir method [1]. Let the function  $F_\lambda^b \in C^\infty(\mathbf{R}^3, \mathbf{R})$ ,  $\lambda \in \mathbf{R}$  given by:

$$\begin{aligned} F_\lambda^b(x_1, x_2, x_3) &= H^b(x_1, x_2, x_3) - \lambda C^b(x_1, x_2, x_3) = \\ &= \frac{b_1}{2} \left( x_2^2 - \frac{b_2}{b_3} x_3^2 \right) - \lambda \left( -\frac{b_3}{2b_1} x_1^2 + x_3 \right). \end{aligned}$$

Then we have successively:

- (i)  $\nabla F_\lambda^b(e_3^m) = 0$  if and only if  $\lambda = \lambda_0$ , where  $\lambda_0 = -mb_1b_2/b_3$ ;
- (ii)  $W := \ker dC_2^b(e_3^m) = \text{span} \left( (1, 0, 0)^T, (0, 1, 0)^T \right)$ ;
- (iii) For all  $v \in W$ , i.e.  $v = (\alpha, \beta, 0)^T$ ,  $\alpha, \beta \in \mathbf{R}$ , we have:

$$v^T \cdot \nabla^2 F_{\lambda_0}^b(e_3^m) \cdot v = \frac{1}{b_1} (-mb_1b_2\alpha^2 + b_1^2\beta^2)$$

and so  $\nabla^2 F_{\lambda_0}^b(e_3^m)|_{W \times W}$  is positive definite (respectively, negative definite) if  $b_1 > 0$  and  $mb_1b_2 < 0$  (respectively,  $b_1 < 0$  and  $mb_1b_2 < 0$ ). Therefore via Arnold's energy-Casimir method we conclude that  $e_3^m$  is nonlinear stable.  $\square$

**Corollary 4.1.** *The equilibrium states  $e_0$  and  $e_1^m, e_3^m$  for  $m \in \mathbf{R}^*$  of the Lorenz-Hamilton system given by (2.3) have the following behavior:*

- (i)  $e_0$  and  $e_1^m$  are nonlinear stable;
- (ii)  $e_3^m$  is nonlinear stable for  $m > 0$  and unstable for  $m < 0$ .

*Proof.* The assertions follows immediately from Propositions 4.2-4.4.  $\square$

## 5 Application to study of an invariant controllable system on $G_4$

Control systems with state evolving on a matrix Lie group arise frequently in physical problems and many others [7, 8].

In this section we present a drift-free left invariant controllable system on a particular Lie group. This arise naturally from the study of the car's dynamics for which the Lie group  $G_4$  represents the configuration space [11].

We denote by  $G_4 \subset UP(4)$  the subgroup of unipotent matrices consisting of elements  $X$  of the form:

$$X = \begin{pmatrix} 1 & x_2 & x_3 & x_4 \\ 0 & 1 & x_1 & \frac{x_1^2}{2} \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (x_1, x_2, x_3, x_4) \in \mathbf{R}^4.$$

A basis of the Lie algebra  $\mathcal{G}_4$  associated to  $G_4$  is  $\{A_1, A_2, A_3, A_4\}$ , where:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = [A_2, A_1], \quad A_4 = [A_3, A_1].$$



Using the results from [7] of controllability for drift-free left invariant systems it follows that there exist only four drift-free left invariant controllable systems on  $G_4$ , namely

$$\dot{X} = X(A_1u_1 + A_2u_2 + pA_3u_3 + qA_4u_4), \quad X \in G_4 \quad (5.1)$$

where  $u_i \in C^\infty(\mathbf{R}, \mathbf{R})$  are control functions and  $(p, q) \in \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ .

An optimal control problem for the system (5.1) with  $(p, q) = (1, 0)$  has been studied in the paper [10].

The space of configurations for kinematics of a car is  $\mathbf{R}^2 \times S^1 \times S^1$ , and its dynamics is described by the system of differential equations :

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = u_1x_2, \quad \dot{x}_4 = u_1x_3. \quad (5.2)$$

The system (5.2) can be interpreted as a drift-free left invariant control system on  $G_4$  [7, 11]. Indeed, the system (5.2) can be written in the equivalent form:

$$\dot{X} = X(A_1u_1 + A_2u_2), \quad \text{where } X \in G_4. \quad (5.3)$$

For the system (5.3) we consider the cost function  $J$  be given by:

$$J(u_1, u_2) = \frac{1}{2} \int_0^{t_f} [c_1u_1^2(t) + c_2u_2^2(t)]dt, \quad c_1 > 0, c_2 > 0. \quad (5.4)$$

Using the Krishnaprasad's theorem [8] we obtain the following proposition.

**Proposition 5.1.** *The controls which minimize the cost function  $J$  given by (5.4) and steers the system (5.3) from  $X(0) = X_0$  at  $t = 0$  to  $X(t_f) = X_f$  at  $t = t_f$  are given by  $u_1 = \frac{z_1}{c_1}$ ,  $u_2 = \frac{z_2}{c_2}$  where  $z_i$ ,  $i = \overline{1, 4}$  are solutions of the system:*

$$\dot{z}_1 = \frac{1}{c_2}z_2z_3, \quad \dot{z}_2 = -\frac{1}{c_1}z_1z_3, \quad \dot{z}_3 = -\frac{1}{c_1}z_1z_4, \quad \dot{z}_4 = 0. \quad \square \quad (5.5)$$

It is easy to see from the equations (5.5) that  $z_4 = k$  ( $k = \text{constant}$ ) and so the system (5.5) can be written in the equivalent form:

$$\dot{z}_1 = \frac{1}{c_2}z_2z_3, \quad \dot{z}_2 = -\frac{1}{c_1}z_1z_3, \quad \dot{z}_3 = -\frac{k}{c_1}z_1. \quad (5.6)$$

We observe that (5.6) is a differential system which belongs to Maxwell-Bloch top system. For the study of geometrical and dynamical properties of the system (5.6) we apply the results given in Sections 2, 3 and 4. To this end we consider the following change of variables:

$$z_1 = y_2, \quad z_2 = y_3, \quad z_3 = y_1. \quad (5.7)$$

Using the relations (5.7), the system (5.6) reads:

$$\dot{y}_1 = -\frac{k}{c_1}y_2, \quad \dot{y}_2 = \frac{1}{c_2}y_1y_3, \quad \dot{y}_3 = -\frac{1}{c_1}y_1y_2. \quad (5.8)$$

Applying Proposition 2.1 for the system (5.8) and the relation (5.7) we obtains the following proposition.

**Proposition 5.2.** *The functions  $\tilde{H}, \tilde{C} \in C^\infty(\mathbf{R}^3, \mathbf{R})$  given by:*

$$\tilde{H}(z_1, z_2, z_3) = -\frac{k}{2c_1} \left( z_1^2 + \frac{c_1}{c_2} z_2^2 \right) \quad \text{and} \quad \tilde{C}(z_1, z_2, z_3) = z_2 - \frac{1}{2k} z_3^2 \quad (5.9)$$

*are constants of the motion for the dynamics (5.6).*  $\square$

**Remark 5.1.** From Proposition 5.2 it follows that *the trajectories of the dynamics (5.6) in the phase space  $\mathbf{R}^3$  are the intersections of the surfaces:*

$$-\frac{k}{c_1} \left( z_1^2 + \frac{c_1}{c_2} z_2^2 \right) = 2\tilde{H}, \quad 2z_2 - \frac{1}{k} z_3^2 = 2\tilde{C},$$

*where  $\tilde{H} = \text{constant}$  and  $\tilde{C} = \text{constant}$ .*  $\square$

Next proposition shows that the dynamics of the system (5.6) reduces to pendulum dynamics.

**Proposition 5.3.** *The solution of the system (5.6), restricted to the constant level surface defined by:*

$$z_1^2 + \frac{c_1}{c_2} z_2^2 = 2H = \text{constant}, \quad H > 0 \quad (5.10)$$

*is*

$$z_1(t) = \sqrt{2H} \cdot \cos \theta(t), \quad z_2(t) = \sqrt{\frac{c_2}{c_1}} \sqrt{2H} \cdot \sin \theta(t), \quad z_3(t) = -\sqrt{c_1 c_2} \cdot \dot{\theta}(t) \quad (5.11)$$

*where  $\theta(t)$  is a solution of the pendulum equation:*

$$\ddot{\theta}(t) = \frac{k}{c_1^2} \sqrt{\frac{c_1}{c_2}} \sqrt{2H} \cdot \cos \theta(t). \quad (5.12)$$

*Proof.* We apply Proposition 2.2 for the system (5.8) and replace  $y_i$  with  $z_j$  according with (5.7).  $\square$

**Proposition 5.4.** *The system (5.6) has the Hamilton-Poisson realization  $(\mathbf{R}^3, \Pi, \tilde{H})$  with the Hamiltonian  $\tilde{H}$  and Casimir  $\tilde{C} \in C^\infty(\mathbf{R}^3, \mathbf{R})$  given by (5.9) and*

$$\Pi(z_1, z_2, z_3) = \begin{pmatrix} 0 & -\frac{1}{k} z_3 & -1 \\ \frac{1}{k} z_3 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

*Proof.* The system (5.8) is a Maxwell-Bloch top system with  $b = (-k/c_1, 1/c_2, -1/c_1)$ .

Applying Proposition 3.1 and the relation (3.2) we obtain that (5.8) has the Hamilton-Poisson realization  $(\mathbf{R}^3, \Pi_1(y), H_1(y))$  with the Casimir  $C_1(y)$ , where

$$\Pi_1(y) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{1}{k} y_1 \\ 0 & \frac{1}{k} y_1 & 0 \end{pmatrix}, \quad H_1(y) = -\frac{k}{2c_1} \left( y_2^2 + \frac{c_1}{c_2} y_3^2 \right), \quad C_1(y) = y_3 - \frac{1}{2k} y_1^2.$$

Replacing in  $H_1(y)$  and  $C_1(y)$  the variables  $y_i$  with  $z_j$  by the change of variables given by (5.7), we find the Hamiltonian  $\tilde{H}(z)$  and Casimir  $\tilde{C}(z)$  given in (5.9).

To determine  $\Pi(z_1, z_2, z_3) = (\{z_i, z_j\})$  we use  $\Pi_1(y)$  and (5.7). We have  $\{z_1, z_2\} = \{y_2, y_3\}_1 = -1/ky_1 = -1/kz_3$ ,  $\{z_1, z_3\} = \{y_2, y_1\}_1 = -1$ ,  $\{z_2, z_3\} = \{y_3, y_1\}_1 = 0$ . Thus we obtain the matrix  $\Pi(z_1, z_2, z_3)$ .  $\square$

The equilibrium states of the dynamics (5.6) are  $\tilde{e}_0 = (0, 0, 0)$ ,  $\tilde{e}_2^m = (0, m, 0)$  and  $\tilde{e}_3^m = (0, 0, m)$  for all  $m \in \mathbf{R}^*$ .

To establish the stability of equilibrium states for the dynamics (5.6) we apply Propositions 4.2-4.4.

**Proposition 5.5.** (i)  $\tilde{e}_0$  and  $\tilde{e}_3^m$  for  $m \in \mathbf{R}^*$  are nonlinear stable.

(ii)  $\tilde{e}_2^m$  for  $m \in \mathbf{R}^*$  are nonlinear stable if  $km > 0$  and unstable if  $km < 0$ .  $\square$

**Conclusions.** In this paper we have presented the geometric and dynamical properties of the Maxwell-Bloch top system (2.1). The denomination used is justified by the fact that the 3D Maxwell-Bloch equations belongs to respective family.  $\square$

**Acknowledgments.** The author has very grateful to be reviewers for their comments and suggestions.

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